

6.1**Slope Fields and Euler's Method****Differential Equations****DEFINITION Differential Equation**

An equation involving a derivative is called a **differential equation**. The **order of a differential equation** is the order of the highest derivative involved in the equation.

EXAMPLE 1 Solving a Differential Equation

Find all functions y that satisfy $dy/dx = \sec^2 x + 2x + 5$.

EXAMPLE 2 Solving an Initial Value Problem

Find the particular solution to the equation $dy/dx = e^x - 6x^2$ whose graph passes through the point $(1, 0)$.

EXAMPLE 3 Handling Discontinuity in an Initial Value Problem

Find the particular solution to the equation $dy/dx = 2x - \sec^2 x$ whose graph passes through the point $(0, 3)$.

EXAMPLE 4 Using the Fundamental Theorem to Solve an Initial Value Problem

Find the solution to the differential equation $f'(x) = e^{-x^2}$ for which $f(7) = 3$.

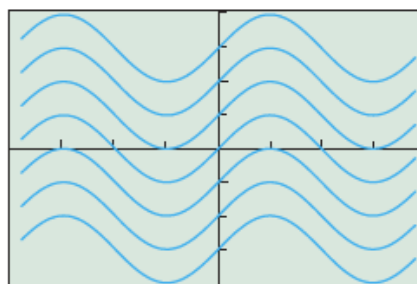
EXAMPLE 5 Graphing a General Solution

Graph the family of functions that solve the differential equation $dy/dx = \cos x$.

SOLUTION

Any function of the form $y = \sin x + C$ solves the differential equation. We cannot graph them all, but we can graph enough of them to see what a family of solutions would look like. The command $\{-3, -2, -1, 0, 1, 2, 3\} \rightarrow L_1$ stores seven values of C in the list L_1 . Figure 6.1 shows the result of graphing the function $Y_1 = \sin(x) + L_1$.

Figure 6.1: Graph of the family of solutions $y = \sin x + C$ for $C \in \{-3, -2, -1, 0, 1, 2, 3\}$.



$[-2\pi, 2\pi]$ by $[-4, 4]$

EXPLORATION 1 Seeing the Slopes

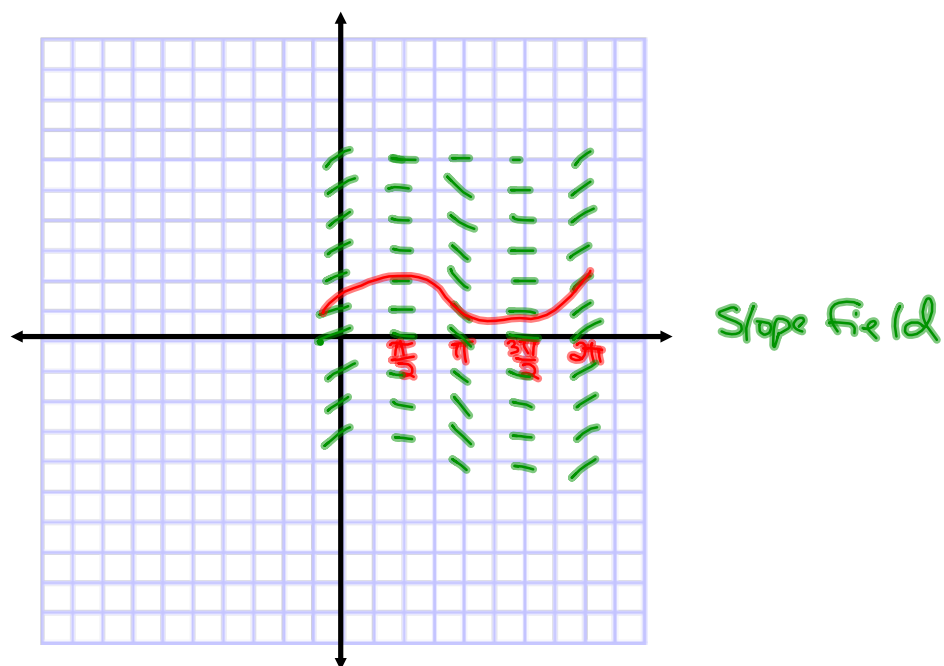
Figure 6.1 shows the general solution to the exact differential equation $dy/dx = \cos x$.

1. Since $\cos x = 0$ at odd multiples of $\pi/2$, we should “see” that $dy/dx = 0$ at the odd multiples of $\pi/2$ in Figure 6.1. Is that true? How can you tell?
2. Algebraically, the y -coordinate does not affect the value of $dy/dx = \cos x$. Why not?
3. Does the graph show that the y -coordinate does not affect the value of dy/dx ? How can you tell?
4. According to the differential equation $dy/dx = \cos x$, what should be the slope of the solution curves when $x = 0$? Can you see this in the graph?
5. According to the differential equation $dy/dx = \cos x$, what should be the slope of the solution curves when $x = \pi$? Can you see this in the graph?
6. Since $\cos x$ is an even function, the slope at any point should be the same as the slope at its reflection across the y -axis. Is this true? How can you tell?

Slope Fields

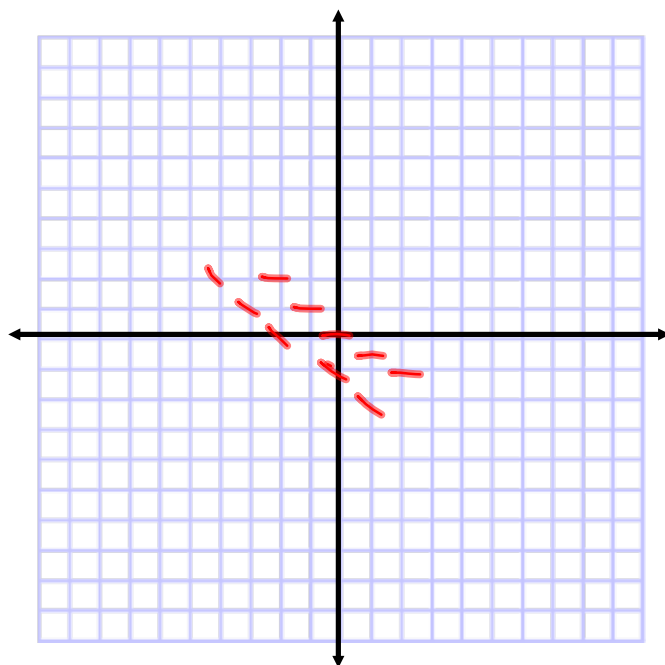
EXAMPLE 6 Constructing a Slope Field

Construct a slope field for the differential equation $dy/dx = \cos x$.



EXAMPLE 7 Constructing a Slope Field for a Nonexact Differential Equation

Use a calculator to construct a slope field for the differential equation $dy/dx = x + y$ and sketch a graph of the particular solution that passes through the point $(2, 0)$.



$$x + y = -1$$
$$y = -x - 1$$

EXAMPLE 8 Matching Slope Fields with Differential Equations

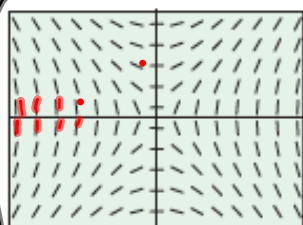
Use slope analysis to match each of the following differential equations with one of the slope fields (a) through (d). (Do not use your graphing calculator.)

1. $\frac{dy}{dx} = x - y$ **b**

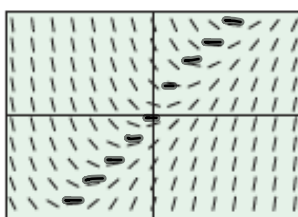
2. $\frac{dy}{dx} = xy$ **d**
 $xy=0$

3. $\frac{dy}{dx} = \frac{x}{y}$ **a**
 undefined when $y=0$
 limit 0 when $x \neq 0$

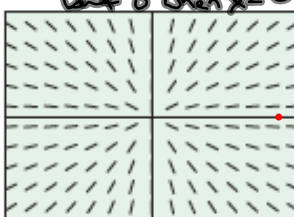
4. $\frac{dy}{dx} = \frac{y}{x}$ **c**



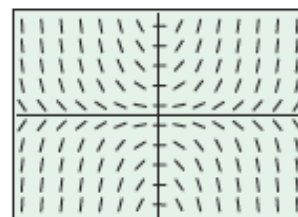
(a)



(b)



(c)



(d)

$x - y = 0$
 $y = x$

$x \rightarrow 0$ slope horizontal
 $y \rightarrow 0$ undefined (vertical)

$\frac{y}{x}$
 $x \rightarrow 0$ slope

Euler's Method

Euler's Method For Graphing a Solution to an Initial Value Problem

1. Begin at the point (x, y) specified by the initial condition. This point will be on the graph, as required.
2. Use the differential equation to find the slope dy/dx at the point.
3. Increase x by a small amount Δx . Increase y by a small amount Δy , where $\Delta y = (dy/dx)\Delta x$. This defines a new point $(x + \Delta x, y + \Delta y)$ that lies along the linearization (Figure 6.6).
4. Using this new point, return to step 2. Repeating the process constructs the graph to the right of the initial point.
5. To construct the graph moving to the left from the initial point, repeat the process using negative values for Δx .

EXAMPLE 9 Applying Euler's Method

Let f be the function that satisfies the initial value problem in Example 6 (that is, $dy/dx = x + y$ and $f(2) = 0$). Use Euler's method and increments of $\Delta x = 0.2$ to approximate $f(3)$.

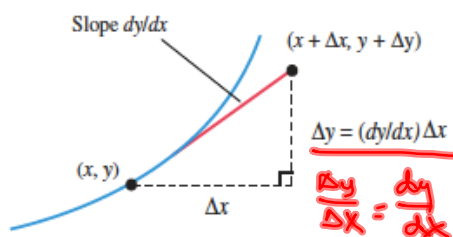


Figure 6.6 How Euler's Method moves along the linearization at the point (x, y) to define a new point $(x + \Delta x, y + \Delta y)$. The process is then repeated, starting with the new point.

(x, y)	$\frac{dy}{dx} = x + y$	Δx	$\Delta y = \left(\frac{dy}{dx}\right)\Delta x$	$(x + \Delta x, y + \Delta y)$
$(2, 0)$	2	.2	.4	$(2.2, .4)$
$(2.2, .4)$	2.6	.2	.52	$(2.4, .92)$
$(2.4, .92)$	3.32	.2	.664	$(2.6, 1.584)$
$(2.6, 1.584)$				

$$(3, 3.46496)$$

$$f(3) \approx 3.465$$

EXAMPLE 10 Moving Backward with Euler's Method

If $dy/dx = 2x - y$ and if $y = 3$ when $x = 2$, use Euler's Method with five equal steps to approximate y when $x = 1.5$.